Math Virtual Learning

AP stats / combinatorics

April 28, 2020

Lesson: April 28, 2020

Objective/Learning Target:

Students will understanding the purpose on combinations and permutations and be able to apply the methods to solve problems.

Review #1

You go to Sonic and want to order from the dollar menu. You can choose a sandwich (either a hamburger or a chicken sandwich), and a side item (fries, tots or salad) and a drink (Pepsi or limeade). How many different meals are possible? You have several tools to solve… you could make a tree diagram, use multiplication, or simply list everything out.

Review #2

In 2019, Danny Duffy struck out 20.7% of the players he pitched against. If we accept this as the probability that he strikes out a runner, what is the probability that he strikes out exactly 3 of the next 5 players?

Answers

1. The easiest method to solve here is to multiply together the number of choices for each option. So, the answer is 2*3*2 = 12 possible meals
2. The best estimate we have here is by applying the binomial setting. We confirm BINS: binary, they strike out or they don’t, Independent, we assume striking one player out does not change probability of striking the next player out (this might not be 100% true), Number, we fix the number of trials at 5, Success, we assume the probability of striking out a player is constant (again, maybe not 100% true, as the batters skill plays into the equation). If we assume these then...

$$p(x = 3) = {}_5C_3\,(0.207)^3(1 - 0.207)^{5-3} \approx 0.0558$$

Combinatorics

Both of these review questions have something in common… they both use an area of mathematics called combinatorics. This is the area of math that is focused on counting the number of ways something can occur. In the first problem, we applied something called the fundamental theorem of counting. In the second, we applied something called a combination… in class we referred to it as the binomial coefficient.

This lesson will help guide you through some basic combinatorics to help you understand the binomial distribution… and hopefully a few ACT questions as well.

Fundamental Theorem of Counting

Fundamental Principle of Counting (FPC): The number of possible outcomes of a series of decisions is found by multiplying the number of choices for each decision.

This is the idea that all of combinatorics is based on. It is also the method used to solve the first review problem. Next, we are going to use it to solve some problems.

How many phone numbers?

Most the phone numbers around here start with area code 816. How many possible phone numbers can be within the 816 area code?

To answer, first remember that leaves us with 7 digits that we can choose a number for. So I write 7 blanks. Each blank can be filled with one of 10 numbers (0-9).

$$\_\ \_\ \_\ \_\ \_\ \_\ \_ = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000000$$

When multiplied, this leaves us with 10,000,000 possible phone numbers. This is convenient as a lot of people have phones.

More complex problems

Ten students are called into the principal's office. They sit outside the office on a bench, waiting. How many arrangements are possible if Bob, Nancy and Trish insist on sitting together?

We use the FPC 3 times to solve this problem…

More complex problems

First, how many ways can Bob, Nancy, and Trish sit together. We have three spaces to fill. The first spot has three options. However, since the same person cannot sit in all three spots, the second spot only has two options. Likewise, the third spot only has 1 option. So…

$$\_\ \_\ \_ = 3 \cdot 2 \cdot 1 = 6$$

Bob, Nancy, and Trish can sit together in a total of 6 ways.

More complex problems

Now, we will consider Bob, Nancy, and Trish as a single unit. Each of the other 7 students are counted individually, so we have 8 units or things we need to now order. Thus,

$$\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_ \quad = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$$

This is the number of ways they can sit together for one way that Bob, Nancy, and Trish can sit next to each other. Since, they can actually sit next to each other in 6 ways we apply the FPC one last time. For an answer of…

$$6 \cdot 40320 = 241,920$$

Factorials

You may have noticed we multiplied all the numbers together between three and 1, and all the numbers together between nine and one. This is a very common operation in combinatorics. So common, that it is given a name, Factorial. It is defined as:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (\ldots) \cdot 1$$

So the previous problem could have been solved as:

$$3! \cdot 8! = 2,177,280$$

Factorials are going to be used whenever we want to obtain the number of ways things can be ordered.

Building a permutation

I am sure we have all been watching a lot of tv lately. So, if we have time for 4 shows, but have 6 in our list of shows to watch… how many ways can we choose four shows to watch. As long as I care about the order I watch the shows, I could use the FPC and get:

$$\_\ \_\ \_\ \_ = 6 \cdot 5 \cdot 4 \cdot 3 = 360$$

This almost looks like a permutation. We can actually use permutations to solve this problem.

Building a permutation

Here is the solution. Can you identify where each piece is coming from?

$$\frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 6 \cdot 5 \cdot 4 \cdot 3 = 360$$

The 6! on top is from the 6 shows we are choosing from. The 6-4 on bottom gives us the number of shows we are not choosing, or the number of left over shows. Simplifying, we see we have the number of shows factorial on top and the number of unwatched shows on bottom factorial. Writing the factorials out as multiplication, we see the 2 and 1 on top and bottom cancel out. Thus we get back to the original answer using the FPC.

Permutation

This process is a permutation. We write it as

Permutation

Permutation of Full Set

$$ {}_{n}P_{k} = \frac{n!}{(n-k)!} $$

Size of Set

\# Selected

Permutation of Left Behind Set

Order?

On a permutation, the order is important! But is order always important?

If we are finding the number of ways to get first, second, and third place in a track event, the runners would probably argue that the order is very important.

If I am choosing three students to be in a group, it doesn't really matter whether you are first or last, just that you are in the group.

How do we find the number of ways when the order does not matter?

Building combinations

We are going to walk through an easy example first. Say I have the letters a,b,c,d. How many ways can I choose 3 of them?

Below we list out each possible permutation. How many are there? What do you notice about each column?

abc	abd	acd	bcd
acb	adb	adc	bdc
bac	bad	cad	cbd
bca	bda	cda	cdb
cab	dab	dac	dbc
cba	dba	dca	dcb

Permutations to Combinations

abc	abd	acd	bcd
acb	adb	adc	bdc
bac	bad	cad	cbd
bca	bda	cda	cdb
cab	dab	dac	dbc
cba	dba	dca	dcb

Each column has the same three elements. So if we wanted combinations, each column would only represent a single value. The order does not matter, so abc is the same as cba which is the same as everything between.

Why are the columns each 6 long? Well 3! = 6

Permutations to Combinations

Each column represents the permutations of the three elements in it, or all the ways those three things can be ordered. We can use this to get the combinations. The general idea here is:

$$\text{combinations} = \frac{\text{permutations of the whole group}}{\text{permutations of each group of size } r} = \frac{{}_nP_r}{r!}$$

For our letters this looks like:

$${}_nC_r = {}_4C_3 = {}_4P_3 \div r! = \frac{4!}{(4-3)!} \div 3! = 4$$

Generalized formula

We can clean up the formula from the last slide a bit shown step by step below.

$$ {}_nC_r = \frac{n!}{(n-r)!} \div r! = \frac{n!}{(n-r)!} \cdot \frac{1}{r!} = \frac{n!}{(n-r)! \cdot r!} $$

The form on the right should look familiar. It is the same equation as the binomial coefficient. In fact they are the same thing! The binomial coefficient is providing the number of ways the series of events in the binomial condition can occur.

Summary

Permutations and Combinations

Number of permutations (order matters) of n things taken r at a time:	Number of combinations (order does not matter) of n things taken r at a time:
$P(n,r) = \frac{n!}{(n-r)!}$	$C(n,r) = \frac{n!}{(n-r)!r!}$

Extra Practice

[Extra Practice]